

# Projective dimension of modules over cluster-tilted algebras

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## Abstract

We study the projective dimension of finitely generated modules over cluster-tilted algebras  $\text{End}_{\mathcal{C}}(T)$  where  $T$  is a cluster-tilting object in a cluster category  $\mathcal{C}$ . It is well-known that all  $\text{End}_{\mathcal{C}}(T)$ -modules are of the form  $\text{Hom}_{\mathcal{C}}(T, M)$  for some object  $M$  in  $\mathcal{C}$ , and since  $\text{End}_{\mathcal{C}}(T)$  is Gorenstein of dimension 1, the projective dimension of  $\text{Hom}_{\mathcal{C}}(T, M)$  is either zero, one or infinity. We define in this article the ideal  $I_M$  of  $\text{End}_{\mathcal{C}}(T[1])$  given by all endomorphisms that factor through  $M$ , and show that the  $\text{End}_{\mathcal{C}}(T)$ -module  $\text{Hom}_{\mathcal{C}}(T, M)$  has infinite projective dimension precisely when  $I_M$  is non-zero. Moreover, we apply the results above to characterize the location of modules of infinite projective dimension in the Auslander-Reiten quiver of cluster-tilted algebras.

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## 1 Introduction

Cluster categories were introduced in [4], and, for type  $A_n$  also in [8], as a means for a better understanding of the cluster algebras of Fomin and Zelevinsky [10,11]. They are defined as orbit categories  $\mathcal{C}_Q = D^b(\text{mod } kQ)/\tau^{-1}[1]$  of the bounded derived categories of finitely generated  $kQ$ -modules, where  $Q$  are acyclic quivers.

In [5], Buan, Marsh and Reiten defined the cluster-tilted algebras as follows. Let  $T$  be a cluster-tilting object in  $\mathcal{C}_Q$ , that is, an object such that  $\text{Ext}_{\mathcal{C}_Q}^1(T, T) = 0$  and the number of non-isomorphic indecomposable summands of  $T$  equals the number of vertices of the quiver  $Q$ . Then the endomorphism algebra  $\text{End}_{\mathcal{C}_Q}(T)$  is called *cluster-tilted algebras*. Since then, these algebras have been the subject of many investigations, see, for instance, [1,2,3,5,6,7,8,9,12].

While cluster categories serve as our main motivation, our main result holds in a more general situation, namely for any triangulated category  $\mathcal{C}$  with a maximal 1-orthogonal object  $T$  (see Definition 2). Generalizing results obtained for cluster-tilted algebras and 2-Calabi-Yau categories in [5,12], Koenig and Zhu show in [13] that the functor  $\text{Hom}_{\mathcal{C}}(T, -)$  induces an equivalence  $\mathcal{C}/\text{add } T[1] \rightarrow \text{mod}(\text{End}_{\mathcal{C}}(T))$ . Moreover, the algebra  $\text{End}_{\mathcal{C}}(T)$  is Gorenstein of dimension 1, thus we know all  $\text{End}_{\mathcal{C}}(T)$ -modules have projective dimension zero, one or infinity. In this paper we characterize the modules of infinite projective dimension.

By the above equivalence, indecomposable  $\text{End}_{\mathcal{C}}(T)$ -modules correspond to indecomposable objects  $M$  in  $\mathcal{C}$  which do not belong to  $\text{add } T[1]$ . For any such object, we denote by  $I_M$  the ideal of  $\text{End}_{\mathcal{C}}(T[1])$  given by all endomorphisms that factor through  $M$  and call it *factorization ideal of  $M$* .

Our main theorem is the following:

**Theorem 1** *Let  $\mathcal{C}$  be a triangulated category with a maximal 1-orthogonal object  $T$ . Let  $M$  be an indecomposable object in  $\mathcal{C}$  which does not belong to  $\text{add } T[1]$ . Then the  $\text{End}_{\mathcal{C}}(T)$ -module  $\text{Hom}_{\mathcal{C}}(T, M)$  has infinite projective dimension precisely when the factorization ideal  $I_M$  is non-zero.*

We discuss in the last section of the paper the fact that the indecomposable  $\text{End}_{\mathcal{C}}(T)$ -modules of infinite projective dimension are lying on certain hammock-like subquivers of the Auslander-Reiten quiver of  $\mathcal{C}$  provided  $\mathcal{C} = \mathcal{C}_Q$  where  $Q$  is Dynkin.

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## 2 Preliminaries

**Definition 2** *Let  $\mathcal{C}$  be a triangulated category. An object  $T$  of  $\mathcal{C}$  is called maximal 1-orthogonal if  $\text{add } T$  is contravariantly finite and covariantly finite, and an object  $X \in \mathcal{C}$  belongs to  $\text{add } T$  if and only if  $\text{Ext}_{\mathcal{C}}^1(X, T) = 0 = \text{Ext}_{\mathcal{C}}^1(T, X)$ .*

Denote by  $C$  the endomorphism ring  $C := \text{End}_{\mathcal{C}} T$ . We recall from [13] that the functor  $\text{Hom}_{\mathcal{C}}(T, -)$  induces an equivalence  $\mathcal{C}/\text{add } T[1] \rightarrow \text{mod } C$ , and moreover, under this equivalence the projective  $C$ -modules are those of the form  $\text{Hom}_{\mathcal{C}}(T, T^0)$  where  $T^0$  belongs to  $\text{add } T$ .

For an object  $M$  in  $\mathcal{C}$ , define  $I_M$  to be the set of all  $f \in \text{End}_{\mathcal{C}} T[1]$  such that there exist morphisms  $g : T[1] \rightarrow M$  and  $h : M \rightarrow T[1]$  such that  $f = h \circ g$ . Clearly,  $I_M$  is an ideal of the algebra  $\text{End}_{\mathcal{C}} T[1] \cong C$ . Since our focus is on the quotient  $\mathcal{C}/\text{add } T[1] \simeq \text{mod } C$ , we will usually consider only objects  $M$  in  $\mathcal{C}$  such that  $M \cap \text{add } T[1] = 0$ . Under these assumptions, it is easy to see that  $I_{M \oplus N} = I_M + I_N$ .

We recall some results that we need later on:

**Lemma 3** ([12]) *Let  $T$  be a maximal 1-orthogonal object in a triangulated category  $\mathcal{C}$ , and let*

$$Y[-1] \longrightarrow X \longrightarrow T^0 \xrightarrow{f} Y$$

*be a triangle in  $\mathcal{C}$  with  $f : T^0 \rightarrow Y$  a contravariant  $\text{add } T$ -approximation of  $Y$ . Then  $X$  belongs to  $\text{add } T$ .*

**Theorem 4** ([13, Thm 2.3]) *Let  $\mathcal{C}$  be a triangulated category with a maximal 1-orthogonal object  $T$ . Let  $g : X \rightarrow Y$  be a morphism in  $\mathcal{C}$  which is a part of a triangle*

$$Z[-1] \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z.$$

*Then  $\text{Hom}_{\mathcal{C}}(T, g)$  is a monomorphism in  $\text{mod } C$  if and only if  $\text{Hom}_{\mathcal{C}}(T, h) = 0$ .*

When there is no danger of confusion, we will sometimes instead of

$$\text{Hom}_{\mathcal{C}}(T, M) \xrightarrow{\text{Hom}_{\mathcal{C}}(T, f)} \text{Hom}_{\mathcal{C}}(T, N)$$

write one of the following simplified forms:

$$\text{Hom}(T, M) \xrightarrow{(T, f)} \text{Hom}(T, N)$$

$$(T, M) \xrightarrow{(T, f)} (T, N).$$

The following theorem has been shown in [12] in the context of 2-Calabi-Yau categories, and in the general context of triangulated categories in [13]:

**Theorem 5 ([13, Cor 4.5])** *Let  $\mathcal{C}$  be a triangulated category and  $T$  a maximal 1-orthogonal object in  $\mathcal{C}$ . Then the endomorphism ring  $C$  of  $T$  is Gorenstein of Gorenstein dimension at most one.*

**Remark 6** *As it has been observed in [12], being Gorenstein of Gorenstein dimension at most one implies that each  $C$ -module has either projective dimension at most one, or is of infinite projective dimension.*

### 3 Proof of the main theorem

Throughout this section, we fix an indecomposable object  $M$  in  $\mathcal{C}$  which is not a direct summand of  $T[1]$ . The aim is to prove that  $\text{Hom}_{\mathcal{C}}(T, M)$  has infinite projective dimension if and only if the factorization ideal  $I_M \neq 0$ . We first show the following implication:

3.1 *The factorization ideal  $I_M$  of  $M$  is non-zero when  $C$ -module  $\text{Hom}_{\mathcal{C}}(T, M)$  has infinite projective dimension.*

**Proof.** Assume  $\text{Hom}_{\mathcal{C}}(T, M)$  has infinite projective dimension, and consider a projective cover

$$\text{Hom}_{\mathcal{C}}(T, T^0) \xrightarrow{(T, f)} \text{Hom}_{\mathcal{C}}(T, M) \longrightarrow 0$$

in  $\text{mod } C$  defined by a morphism  $f : T^0 \rightarrow M$  in  $\mathcal{C}$ . Choose a triangle in  $\mathcal{C}$

$$M[-1] \xrightarrow{\beta[-1]} T^1 \longrightarrow T^0 \xrightarrow{f} M$$

containing  $f$ , then by Lemma 3 we know that  $T^1$  belongs to  $\text{add } T$ . Applying the functor  $(T, -) := \text{Hom}_{\mathcal{C}}(T, -)$  to that triangle yields an exact sequence in  $\text{mod } C$

$$(T, M[-1]) \xrightarrow{(T, \beta[-1])} (T, T^1) \longrightarrow (T, T^0) \longrightarrow (T, M) \longrightarrow (T, T^1[1])$$

where  $\text{Hom}_{\mathcal{C}}(T, T^1[1]) = 0$  since  $T$  is a tilting object. This implies that the morphism  $\text{Hom}_{\mathcal{C}}(T, \beta[-1])$  is non-zero, since otherwise the projective dimension of  $\text{Hom}_{\mathcal{C}}(T, M)$  would be at most one. Choose a morphism  $\alpha[-1]$  in  $\text{Hom}_{\mathcal{C}}(T, M[-1])$  whose image under  $\text{Hom}_{\mathcal{C}}(T, \beta[-1])$  is non-zero, that is, the

composition

$$T \xrightarrow{\alpha[-1]} M[-1] \xrightarrow{\beta[-1]} T^1$$

is non-zero. This yields the non-zero composition

$$T[1] \xrightarrow{\alpha} M \xrightarrow{\beta} T^1[1]$$

As  $T^1[1]$  is a non-trivial summand in  $\text{add } T[1]$ , we conclude that there is a non-zero element in the factorization ideal  $I_M$  of  $M$ .  $\square$

*3.2 The factorization ideal  $I_M$  of  $M$  is non-zero only if  $\text{Hom}_{\mathcal{C}}(T, M)$  has infinite projective dimension.*

**Proof.** By remark 6, a  $\mathcal{C}$ -module  $\text{Hom}_{\mathcal{C}}(T, M)$  of finite projective dimension has projective dimension zero or one. It suffices to show that in both cases the ideal  $I_M$  is zero.

**Case 1:** Assume  $\text{Hom}_{\mathcal{C}}(T, M)$  has projective dimension 0.

Then  $M$  belongs to  $\text{add } T$ , and every composition of morphisms

$$T[1] \xrightarrow{\alpha} M \xrightarrow{\beta} T[1]$$

must be zero since  $\beta \in \text{Hom}_{\mathcal{C}}(M, T[1]) = 0$ . Therefore  $I_M = 0$  in this case.

**Case 2:** Assume  $\text{Hom}_{\mathcal{C}}(T, M)$  has projective dimension 1.

Thus there is a projective resolution

$$(1) \quad 0 \longrightarrow (T, T^1) \xrightarrow{(T, g)} (T, T^0) \longrightarrow (T, M) \longrightarrow 0$$

in  $\text{mod } \mathcal{C}$ . We choose a triangle

$$(2) \quad Z[-1] \xrightarrow{h} T^1 \xrightarrow{g} T^0 \xrightarrow{f} Z$$

in  $\mathcal{C}$ , and by theorem 4 we conclude that  $\text{Hom}_{\mathcal{C}}(T, h) = 0$ . Since moreover  $\text{Hom}_{\mathcal{C}}(T, T^1[1]) = 0$ , applying  $(T, -) = \text{Hom}_{\mathcal{C}}(T, -)$  to the triangle (2) yields a short exact sequence in  $\text{mod } \mathcal{C}$

$$(3) \quad 0 \longrightarrow (T, T^1) \xrightarrow{(T, g)} (T, T^0) \xrightarrow{(T, f)} (T, Z) \longrightarrow 0.$$

Since the two short exact sequences (1) and (3) start with the same morphism  $(T, g)$ , we conclude that their cokernels  $\text{Hom}_{\mathcal{C}}(T, M)$  and  $\text{Hom}_{\mathcal{C}}(T, Z)$  are isomorphic. This implies in the category  $\mathcal{C}$  that the objects  $M$  and  $Z$  differ only

by summands in  $\text{add } T[1]$ . But we assumed  $M$  to be indecomposable and not isomorphic to an object in  $\text{add } T[1]$ , hence  $M$  is isomorphic to a summand of  $Z$ . We denote by  $\iota : M \rightarrow Z$  the corresponding section with retraction  $\rho : Z \rightarrow M$ .

Given a factorization

$$T[1] \xrightarrow{\alpha} M \xrightarrow{\beta} T[1]$$

the aim is to show that  $\beta\alpha = 0$  and hence  $I_M = 0$ . We consider the composed maps

$$T[1] \xrightarrow{\iota\alpha} Z \xrightarrow{\beta\rho} T[1]$$

and insert them into a commutative diagram formed using the triangle (2):

$$(4) \quad \begin{array}{ccccccc} & & T & \longrightarrow & 0 & \longrightarrow & T[1] \xrightarrow{\mathbb{I}} T[1] \\ & & \downarrow x & & \downarrow & & \downarrow \iota\alpha \\ Z[-1] \xrightarrow{h} & T^1 & \xrightarrow{g} & T^0 & \xrightarrow{f} & Z & \longrightarrow T^1[1] \\ & \downarrow y & & \downarrow & & \downarrow \beta\rho & \\ & T & \longrightarrow & 0 & \longrightarrow & T[1] \xrightarrow{\mathbb{I}} T[1] \end{array}$$

The existence of a morphism  $x$  making the upper part of the diagram commutative is guaranteed by the following commutative square between two triangles in  $\mathcal{C}$ :

$$\begin{array}{ccc} 0 & \longrightarrow & T[1] \\ \downarrow & & \downarrow \iota\alpha \\ T^0 & \xrightarrow{f} & Z \end{array}$$

Likewise, the existence of a morphism  $y$  making the lower part of the diagram commutative is guaranteed by the following square

$$\begin{array}{ccc} T^0 & \xrightarrow{f} & Z \\ \downarrow & & \downarrow \beta\rho \\ 0 & \longrightarrow & T[1] \end{array}$$

which commutes since  $\text{Hom}_{\mathcal{C}}(T^0, T[1]) = 0$ . From the commutative square

$$\begin{array}{ccc} T & \longrightarrow & 0 \\ \downarrow x & & \downarrow \\ T^1 & \xrightarrow{g} & T^0 \end{array}$$

in (4) we conclude that  $gx = 0$ , and hence  $x = 0$  since  $\text{Hom}_{\mathcal{C}}(T, g)$  is a monomorphism. This implies  $x[1] = 0$  and therefore, by the commutativity of (4),

$$0 = y[1]x[1] = \beta\rho\alpha = \beta\alpha$$

which implies  $I_M = 0$ .

□

**Remark 7** We can apply the dual proof to obtain that the  $\text{End}_{\mathcal{C}}T$ -module  $\text{Hom}_{\mathcal{C}}(T, M)$  has infinite injective dimension precisely when the factorization ideal  $I_M$  is non-zero. However, it is easy to see for any Gorenstein algebra that the modules of infinite projective dimension are exactly the modules of infinite injective dimension, so it is no surprise they satisfy the same condition.

**Remark 8** We showed in the proof of case 2 in 3.2 that there exists the triangle (2) in  $\mathcal{C}$  where  $M$  and  $Z$  differ only by summands in  $\text{add } T[1]$ , that is  $Z \cong M \oplus Z'$  where  $Z' \in \text{add } T[1]$ . We then continue to show in (4) that there is no morphism from  $T[1]$  to  $T[1]$  factoring through  $Z$ . This implies that  $Z' = 0$ , hence we can extract from the proof above the following lifting property:

**Corollary 9** *Every short exact sequence*

$$(1) \quad 0 \longrightarrow (T, T^1) \xrightarrow{(T, g)} (T, T^0) \xrightarrow{(T, f)} (T, M) \longrightarrow 0$$

*in  $\text{mod } \mathcal{C}$  with  $T^0, T^1 \in \text{add } T$  can be lifted to a triangle*

$$(2) \quad M[-1] \xrightarrow{h} T^1 \xrightarrow{g} T^0 \xrightarrow{f} M$$

*in  $\mathcal{C}$ .*

## 4 Hammocks

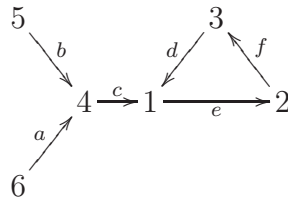
We discuss in this section how to compute the objects  $M \in \mathcal{C}$  with  $I_M \neq 0$ . Of course one has  $I_M \neq 0$  if and only if there are indices  $i, j$  such that there exists a non-zero morphism between the indecomposable summands  $T_i[1]$  and  $T_j[1]$  of  $T[1]$  factoring through  $M$ . Define the following full subquiver

$$H(i, j) = \{X \in \mathcal{C} \mid \text{there is } 0 \neq \lambda : T_i[1] \rightarrow T_j[1] \text{ factoring through } X\}$$

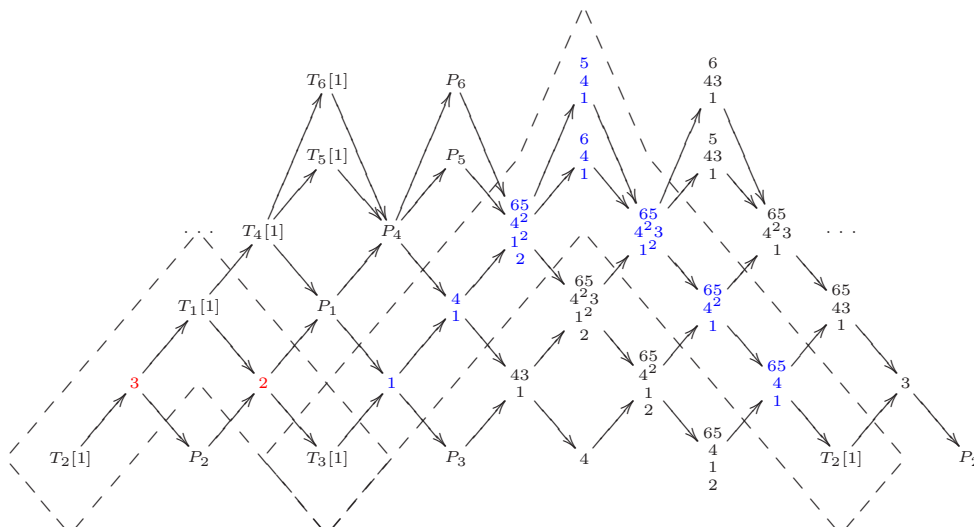
of the Auslander-Reiten quiver of  $\mathcal{C}$ . Our main theorem implies that the functor  $\text{Hom}(T, -)$  induces a bijection between the set

$$\bigcup_{i, j} (H(i, j) \setminus T[1])$$

**Example:** Let  $C$  be the cluster-tilted algebra given by the quiver



It turns out that only for the three pairs  $(i, j) = (2, 1), (1, 3), (3, 2)$  the set  $H(i, j) \setminus T[1]$  is non-empty (there are no  $C$ -modules lying on a path from, say,  $T_4[1]$  to  $T_6[1]$ ). The modules in  $H(3, 1)$  are indicated in the figure below in blue, whereas the modules in  $H(2, 1)$  and  $H(1, 3)$  are red. One observes that in the Dynkin-case where all Hom-dimensions are zero or one, the subquivers  $H(i, j)$  are hammocks.





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